# (co)fiber sequences and $\pi_3(S^2)$

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This note attempts to give an intuitive explanation of cofiber and fiber sequences in as basic of a language as possible without losing rigor[1]. Then the classical application of the fiber sequence induced long exact sequence to the calculation of $\pi_3(S^2)$ is justified. In essence this is a spelled out guide for sections 8.1-5 and 9.3 of May's A Concise Course in Algebraic Topology.

## 1 Motivations

We want to study the structures of new spaces. And one obvious way to go about this is to study maps between spaces that we know a lot about and these unknown spaces.

But an arbitrary map can behave very badly even if we assume strong conditions like continuity. One such example is the space filling curves.

Therefore we really want to work with nice maps that carry over properties and knowledge we desire. Intuitively, the simplest of maps are inclusions/injections and projections/surjections. As a result, given an arbitrary map, we would like to compare how similar it is to suitable notion of injective maps and surjective maps[2] in a nice category of spaces.

## 2 Based Spaces and Based Maps

The category of spaces that we will be working in are the based spaces. They are nice topological spaces[3] with an added distinguished point called its base point (from now on denoted by $*$), like how a group has the identity as a distinguished element.[4]

We call maps between based spaces *based maps* if it is continuous and respects the base point, i.e. $f : X \to Y$ is based if $f(*_X) = *_Y$.

Luckily the function space of based maps $\mathrm{Maps}(X, Y)$ for $X, Y$ based spaces is also a based space with its base point being the constant map sending everything to base point of $Y$. [5]

Turns out we can have a natural adjunction homeomorphism[6]

$$\mathrm{Maps}(X, \mathrm{Maps}(Y, Z)) \cong \mathrm{Maps}(X \wedge Y, Z) \tag{1}$$

We know what is on the left hand side of the equation, and to understand the right-hand side we only need to know what does $\wedge$ mean.

[1] the (not) notorious "explain like I'm an analyst" style, no disrespect to analysis.

[2] cofibrations and the (Hurewicz) fibration.

[3] Specifically the compactly generated topological spaces. See chapter 5 of May.

[4] Our category is pointed, has a zero object, namely the space with just the base point.

[5] This means that based maps of spaces are exponential, or our special category is cartesian closed.

[6] recognizable as an analog of people's favorite tensor-hom adjunction, namely $Hom(X, Hom(Y, Z)) \cong Hom(X \wedge Y, Z)$, except the homomorphisms and products are the ones in our special category.

**Definition.** $X \wedge Y := (X \times Y)/(X \vee Y)$[7]

Here $\times$ denote the usual Cartesian product with base point $(*_X, *_Y)$. And $(X \vee Y)$ is space $X, Y$ glued at their base point with no other changes, i.e. $(*_X, y) \cup (x, *_Y)$ for all $x \in X, y \in Y$ with base point also at $(*_X, *_Y)$. And the thing behind the slash is the quotient, in other words, they are squeezed together to form the base point of the resulting space.

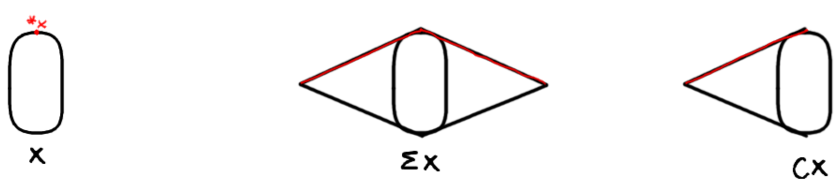
Now to justify our relation 1, we only need to verify that base points are matched correctly on both sides because continuity should be preserved trivially.

Another important property of smash product is that it is also (anti)symmetric with respect to the two input spaces. In other words, $X \wedge Y = -Y \wedge X \cong X \wedge Y$.

**Definition.** Below $I$ is the unit interval $[0, 1]$ and $S^1$ is the circle[8]

1. The loop space of X is $\Omega X := \mathrm{Maps}(S^1, X)$, with base point the constant loop about $*_X$.
2. The (reduced) suspension of X is $\Sigma X := X \wedge S^1$, with base point indicated in red below.
3. The path space[9] of X is $PX := \mathrm{Maps}(I, X)$, again with base point the constant map of $*_X$.
4. The (reduced) cone of X is $CX := (X \times I)/(X \times \{0\})$, with base point indicated in red below.

Here are some pictures illustrating the above definitions. From now on we denote the base point in color red.




**Proposition 2.1.** *Clearly substituting $S^1$ in place of $Y$ in Equation 1, we get*

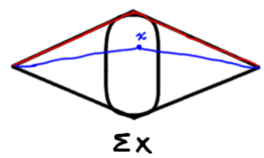
$$Maps(X, \Omega Z) \cong Maps(\Sigma X, Z) \tag{2}$$

This is evidently true even without resorting to our previous adjunction relation. Given a point $x \in X$, we canonically picks out a loop in $\Sigma X$. Thus mapping $x$ to loops in some space $Z$ is really just mapping a loop in $\Sigma X$ to $Z$. A picture of the loop in $\Sigma X$ given a point $x \in X$ is shown below.

[7] It was brought to my attention that the choice of notation has to do with how a cross $\times$ is formed by a $\vee$ sitting on top of a $\wedge$, fulfilling the usual intuition of quotients.

[8] Turns out homotopically, the contractible spaces and the circle are two spaces that we know really well.

[9] Old literature adhereing to Serre might use the notation $EX$. This is also distinguished from $Y^I$ in May as the paths in $PX$ have to start at the base point while the other one does not (Hence it is appropriatedly named the "free path space"). This is important as $PX$ is clearly contractible by tracing back to the starting point of each path, while one cannot do that in $X^I$.




## 3 Cofiber and Fiber Sequences

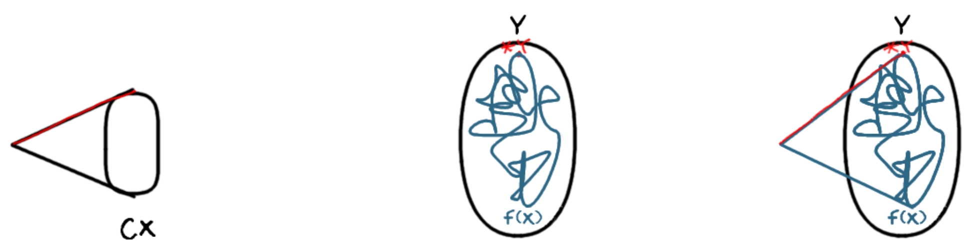
Now we want to construct our main players - the cofiber and fiber sequences. The cofiber sequence gives us concrete information on how related is our arbitrary map $f : X \to Y$ is to a nice inclusion, and the fiber sequence to a nice surjection.

**Definition.** The cofiber sequence of map $f : X \to Y$ is

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \Sigma Cf \longrightarrow \Sigma\Sigma X \longrightarrow \cdots \tag{3}$$

where $i$ is an inclusion map and $Cf$ is called the (homotopy) cofiber of the map $f$. The sequence continues indefinitely to the right with one more suspension operator every 3 steps.

The cofiber $Cf$ can be defined as $CX \cup_f Y$. This notation means that we glue the cone of $X$ and $Y$ along the image of $X$ under $f$. Pictorially,




We can also express this relation via a (commuting) pushout square:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "(id,1)"'] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "i"] \\ CX \arrow[r] & X \cup_f Y \end{tikzcd}$$

Crucially, we are calling the above expression 3 the cofiber sequence because (homotopically) we are doing the same thing to our map every step.

**Lemma 3.1.** $\Sigma X \simeq Ci$

*Proof.* Using our definition of the cofiber above, we know $Ci$ is $CY$ glued to the image of $Y$ in $Cf$. Now this give us a homotopical way to contract things outside of $f(X)$ to the base point because we can just "move" the line of base points outside. Therefore we get $\Sigma f(X)$. Since copies of $X$ lives in between the pinched ends and $f(X)$ in the middle, it really is just $\Sigma X$.


□

Based on our previous adjuction relation, we can in fact obtain a "dual" sequence.

**Definition.** The fiber sequence of map $f : X \to Y$ is

$$\cdots \longrightarrow \Omega\Omega Y \longrightarrow \Omega Ff \longrightarrow \Omega X \longrightarrow \Omega Y \longrightarrow Ff \xrightarrow{\pi} X \xrightarrow{f} Y \tag{4}$$

where $\pi$ is a projection map and $Ff$ is called the (homotopy) fiber of the map $f$. The sequence continues indefinitely to the left with one more looping operator every 3 steps.

The fiber $Ff$ is defined to be $X \times_f PY$[10]. This notation means that $X \times_f PY = \{(x, \gamma) \in X \times PY : f(x) = \gamma(1)\}$.

The same information is conveyed in the (commuting) pullback square below.

$$\begin{tikzcd}
X \times_f PY \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner"] & PY \arrow[d, "ev(1)"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

We are calling expression 4 fiber sequence because (homotopically) we are doing the same thing to our map every step as well.

**Lemma 3.2.** $\Omega Y \simeq F\pi$

*Proof.* Applying the definition of the fiber to the map $\pi$, we get $F\pi = \{((x, \gamma), \alpha) \in Ff \times PX : \pi(x, \gamma) = x = \alpha(1)\} = \{(x, \gamma, \alpha) \in X \times PY \times PX : x = \alpha(1), f(x) = \gamma(1)\}$. We can see this is equivalent to $\Omega Y$ because a coordinate $(x, \gamma, \alpha)$ contains exactly the same data as that of a loop in Y. Namely we have two paths $f(\alpha)$ and $\gamma$ starting at the base point and ends at $f(x)$, which forms a loop. See picture below.

[10] the fibered product


$\square$

# 4 Long Exact Sequences of Homotopic Maps

**Definition.** The homotopy classes of maps from space $X$ to $Y$ is defined as

$$[X, Y] := \mathrm{Maps}(X, Y)/\simeq$$

**Theorem 4.1.** *For a fiber sequence of $f : X \to Y$ and an arbitrary based space $Z$, we have the following a long exact sequence (LES) of maps up to homotopy:*

$$\cdots \longrightarrow [Z, \Omega Ff] \longrightarrow [Z, \Omega X] \longrightarrow [Z, \Omega Y] \longrightarrow [Z, Ff] \xrightarrow{\pi^*} [Z, X] \xrightarrow{f^*} [Z, Y]$$

*Here a LES means that at successive stage* $Z' \xrightarrow{i} Z \xrightarrow{j} Z''$ *we have $j(z) = *_{Z''}$ if and only if there is a $z'$ such that $z = i(z')$.* [11]

*Proof.* We separate the proof into 3 parts.

1. We claim that the last stage of the sequence is exact. This stage is exact if (a) for any map $g : Z \to X$ with the property $f^*(g)$ is trivial, there is $g' : Z \to Ff$ such that $\pi^*(g') = g$, and (b) $f^*(\pi^*(g'))$ is trivial for all $g' : Z \to Ff$. Note that by definition, $f^*(g)(z) = (f \circ g)(z)$, thus it is trivial means that $f(g(z))$ is homotopic to the trivial map. Since $PY$ is contractible, the said homotopy gives a map from $Z$ to $PY$. Thus by universal property of pullback squares, we know there exists a suitable $g'$. For (b) we know $f^*(\pi^*(g')) = (f \circ \pi \circ g')$ which is homotopic to the trivial map since we can compose $g'$ with the projection to the contractible $PY$.

$$\begin{tikzcd} Z \arrow[r, dotted, "g'"] \arrow[rr, bend left] \arrow[dr, "g"'] & Ff \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\lrcorner"] & PY \arrow[d, "ev(1)"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

[11]One motivation for this exactness criterion is that it is sort of the easiest chain of maps we can think of; anything goes further than a stage is *exactly* the trivial map. One imprecise analogy can be made between this and the memoryless-ness of Markov chains, where states of $\geq 2$ steps ago have trivial/no effects on the current state.

2. Looping and fibering commutes[12].

   First we need to show $\Omega(PY) \cong P(\Omega Y)$. By passing through the adjunction, we can use the (anti)symmetry of smash products between $I$ and $S^1$ and then back through the adjunction relation to get it.

   Because looping is a functor, we can apply it to our previous pullback square and retain it's universal property.

$$\begin{tikzcd} \Omega(Ff) \arrow[r] \arrow[d, "\Omega\pi"] \arrow[dr, phantom, "\lrcorner"] & \Omega(PY) \arrow[d] \\ \Omega X \arrow[r, "\Omega f"] & \Omega Y \end{tikzcd}$$

   But then using $\Omega(PY) \cong P(\Omega Y)$ at the upper right corner and by definition of the (homotopy) fiber, we get another pullback square,

$$\begin{tikzcd} F(\Omega f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & P(\Omega Y) \arrow[d] \\ \Omega X \arrow[r, "\Omega f"] & \Omega Y \end{tikzcd}$$

   Thus by universal property of pullbacks, we have $\Omega(Ff) \cong F(\Omega f)$.

3. Since $PY$ is contractible to the constant path at $*_Y$, we can "extend" this homotopy through our diagram to get a homotopy between $Ff = X \times_f PY$ and $X \times_f cst(*_Y) = \{x \in X : f(x) = *_Y\} = f^{-1}(*_Y)$[13].

4. Finally, show the triangle and induct, we can prove that the whole sequence is exact.

□

We have a corresponding theorem about LES for the cofiber sequence. The only difference is that there we are mapping from our (cofiber) sequence of spaces to $Z$ an arbitrary space[14]. The proofs are almost exact parallels of above (also contained in May section 8.4).

To make everything more practical, we want to translate this LES of based spaces to that of abelian groups, a category where everything is classified and exactness reduce to the normal notion of ker = im.

**Proposition 4.2.** $[\Sigma\Sigma X, Y]$ *is an abelian group for all based spaces $X$ and $Y$.*

This can be reasoned as commuting two small (homotopy) squares inside a big (homotopy) square by selectively contracting parts to the constant (base point) map. It is done at the end of May section 8.2.

[12] This in fact generalizes to limits and limits or colimits and colimits commute.

[13] While this is a correct reasoning, a rigorous proof would require the explicit production of that "extended" homotopy, which is presented as a lemma in the Appendix.

[14] Contravariant instead of covariant.

## 5 Facts about Circles and Spheres

First a fact that is easy to believe:

**Proposition 5.1.** $\Sigma S^n = S^{n+1}$

In low dimensions, a simple picture like the ones in secion 2 above will do. In general, we can see that $S^n$ is exactly the equator of $S^{n+1}$.

Next we introduce the concept of "Hopf bundle" or "Hopf fibration." These are maps between a source and a target sphere of certain dimensions with sphere of another lower dimension as its fiber. It is a (fiber) bundle in the sense that on any open set of the base sphere, the source looks like Cartesian product of the fiber and the target[15]. The one of interest to us is the following

$$\begin{tikzcd} f^{-1}(pt) \cong S^1 \arrow[r, hook] & X = S^3 \arrow[d, two heads, "f"] \\ & Y = S^2 \end{tikzcd}$$

To be able to apply our fiber sequence machinery to the Hopf map, we need to know that up to homotopy, the actual fiber ($f^{-1}(pt) \cong S^1$) from this fiber bundle is the same as the fiber ($Ff$) that was used in the fiber sequence earlier[16].

**Proposition 5.2.** $[S^m, S^m] \cong \mathbb{Z}$ *for all* $m \in \mathbb{Z}^{\geq 1}$.

This homeomorphism is realized by the degree map. Intuitively we can see that wrapping spheres around itself different number of times should give non homotopic maps as the preimage of any point changes multiplicity.

**Proposition 5.3.** $[S^k, S^1] \cong 0$ *for all* $k \neq 1$.

This is true because the universal cover $\mathbb{R}$ of $S^1$, realized by the map $\phi : \mathbb{R} \to S^1 \subset \mathbb{C}$ by sending $t$ to $\exp(2\pi\sqrt{-1}t)$, is contractible. Then by covering space theory, the lifting property[17] dictates that maps into $S^1$ can be lifted to maps into $\mathbb{R}$. Thus contractibility gives a homotopy of any map to the constant trivial map.

## 6 Punchline

**Theorem 6.1.** $\pi_3(S^2) = [S^3, S^2] \cong \mathbb{Z}$

[15] This property is usually called 'local triviality.' The existence of such Hopf bundles at the specified dimensions are closely related to the "uniqueness" of the 4 division algebras, namely $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and their corresponding unit "norm" spheres.

[16] This require first knowing Hopf bundle is a (Hurewicz) fibration, then applying the lemma in Appendix again gives us what we want. Turns out that in general, a fiber bundle does not have to be a (Hurewicz) fibration.

[17] fibration and CHP

*Proof.* Inspecting the LES induced by the fiber sequence of the Hopf bundle (i.e. a substitution of the total space $E = S^3$ for $X$, base space $B = S^2$ for $Y$, fiber bundle map $p$ for $f$, fiber $F = S^1$ for $Ff$, and finally the space $Z = S^2 = \Sigma\Sigma S^0$), we get the following exact sequence in abelian groups

$$\cdots \longrightarrow [S^2, \Omega S^1] \longrightarrow [S^2, \Omega S^3] \longrightarrow [S^2, \Omega S^2] \longrightarrow [S^2, S^1] \longrightarrow \cdots$$

Applying the adjunction relation (Eq. 2) to first three terms and using proposition 5.2 on the first and last term and proposition 5.3 on the second term, we get a "short exact sequence"

$$[S^2, \Omega S^1] \cong [S^3, S^1] \cong 0 \longrightarrow [S^2, \Omega S^3] \cong [S^3, S^3] \cong \mathbb{Z} \longrightarrow [S^2, \Omega S^2] \cong [S^3, S^2] \longrightarrow [S^2, S^1] \cong 0$$

Therefore the injective-ness and surjective-ness of the middle map ensures that $[S^3, S^2] \cong \mathbb{Z}$. □

This theorem shows that there is somehow a whole integral family of nontrivial maps from the 3-sphere to the 2-sphere up to homotopy, a fact that topologists would sometimes analogize to the 2-sphere having a "3-dimensional hole."

## Acknowledgments

Many thanks to Prof. Andrew Blumberg and Ernest Fontes for putting up with my endless cascade of silly questions.

# Appendix

**Lemma.** (cf. exercise #1 in Chapter 8 of May) For a fibration $f : X \to Y$, we have $Ff \simeq f^{-1}(*_Y)$

*Proof.* Since we have a commuting square, the fibration gives us a lifting map

$$\begin{tikzcd} Ff = X \times_f Y^I \arrow[r, "\pi"] \arrow[d, hook, "(id,id,1)"'] & X \arrow[d, two heads, "f"] \\ Ff \times I \arrow[r, "g"] \arrow[ur, dotted, "\tilde{g}"] & Y \end{tikzcd}$$

where $g(x, \gamma, t) = \gamma(t)$ and $\pi$ is the projection on first coordinate.

Next we can construct a continuous homotopy $H_t : Ff \to Ff$ via $H_t(x, \gamma) = (\tilde{g}(x, \gamma, t), \gamma|_{[0,t]})$. Note that when $t = 1$, we have $H_1(x, \gamma) = (x, \gamma)$ the identity map on $Ff$. When $t = 0$, it gives $H_0(x, \gamma) = (f^{-1}(\gamma(0) = *_Y), cst_{*_Y})$.

Now define $j : f^{-1}(*_Y) \to Ff$ via $j(x) = (x, cst_{*_Y})$, then the above gives the following diagram:

$$\begin{tikzcd} Ff \arrow[r, "\tilde{g}(-,-,0)"] \arrow[rr, bend right, "H_1 \simeq id"'] & f^{-1}(*_Y) \arrow[r, "j"] & Ff \end{tikzcd}$$

This demonstrates that we have a homotopic left inverse of $j$. Since we have projection onto the first coordinate as a right inverse, namely $\pi \circ j(x \in f^{-1}(*_Y)) = \pi(x, cst_{*_Y}) = x$, we have proved that $Ff \simeq f^{-1}(*_Y)$. $\square$